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LETTER TO THE EDITOR

Renormalisation-group study of fully directed self-avoiding walks†

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Abstract. We present a discussion of the critical properties of fully directed self-avoiding walks on hypercubic lattices. A position-space renormalisation group transformation is carried out analytically in two dimensions for any value of the scaling factor b; the finite-size scaling properties of the transformation are explicitly exhibited, including the well known finite-size scaling hypothesis $\nu(b)-\nu \sim (\ln b)^{-1}$. It is shown that the RG transformation in this case is sensitive to $\nu_{\|}$, thus giving support to existing arguments.

In the past few years, a considerable amount of work has been done in the study of directed lattice problems: to give only a partial list of references, we note that directed percolation has been studied through finite-size scaling (Domany and Kinzel 1981, Kinzel and Yeomans 1981) and position-space renormalisation group (PSRG) techniques (Redner 1981, 1982, Redner and Brown 1981, Oliveira 1983); the concept of pseudo-correlation length has been discussed by Klein and Kinzel (1981); directed lattice animals were studied by Redner and Yang (1982), Redner and Coniglio (1982) and Family (1982), and their relation to the Lee–Yang edge singularity has been discussed by Cardy (1982) and Stanley *et al* (1982); Chakrabarti and Manna (1983) introduced a type of directed self-avoiding walk (SAW), whose properties they established mainly through the analysis of numerical data. Directed SAWS are studied also by Redner and Majid (1983) through transfer-matrix techniques, and by Cardy (1983), who uses a field-theoretical approach.

In this letter we present a discussion of critical properties of fully directed SAWS on hypercubic lattices. Besides its interest as a problem on lattice statistics in its own right, the possibility arises that directed SAWS may be of use as a model for linear polymers under anisotropic conditions, e.g. in a flowing solvent.

We first define the problem and recall some exact results; then we show that a PSRG calculation for fully directed SAWS on a square lattice can be carried out exactly for any scaling factor and discuss the limiting behaviour of the RG transformation as the scaling factor approaches infinity; finally we compare our results with those of Chakrabarti and Manna (1983) and briefly summarise our findings.

Consider a d-dimensional hypercubic lattice and the geometrical object that is built by starting at a given point (the origin) and advancing at random through steps linking nearest-neighbour sites, with the constraint that any step be given only in the positive direction of a cartesian axis. This is what we define as a fully directed SAW. Undirected

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SAWs have long since been used as a model for linear polymers in good, isotropic, solvents (McKenzie 1976, de Gennes 1979); of course, the constraint on directionality already implies non-intersection, but the converse is not true: directionality, in the sense stated above, is a much more stringent condition than non-intersection. Actually, directed and undirected SAWs belong to different universality classes, as already pointed out by Chakrabarti and Manna (1983).

As usual in directed problems, two distinct lengths diverge in the limit of large number N of steps: $\xi_\parallel$, which gives the average radius of the walk parallel to the $(1, 1, \ldots, 1)$ or 'easy' axis, and $\xi_\perp$, related to the transverse width in directions perpendicular to the anisotropy axis. In analogy with the isotropic case, we define two 'correlation length' exponents $\nu_\parallel$ and $\nu_\perp$ through

$$\xi_\parallel \sim N^{\nu_\parallel}, \qquad \xi_\perp \sim N^{\nu_\perp}, \qquad N \to \infty. \tag{1}$$

For any dimension d, every directed step gives a positive contribution of the same size to $\xi_\parallel$, hence

$$\nu_\parallel = 1 \tag{2}$$

for any d. On the other hand, the projection of the directed SAW on the perpendicular direction is a random walk, whence

$$\nu_\perp = \tfrac{1}{2} \tag{3}$$

for any d.

Another results can be obtained if we analyse the susceptibility exponent γ; we note that the number of fully directed SAWs of n steps on a d-dimensional hypercubic lattice is

$$N_n = d^n. \tag{4}$$

Since the exponent γ is defined through

$$N_n \sim \mu^n n^{\gamma - 1} \tag{5}$$

where μ is the effective coordination number of the lattice (McKenzie 1976, de Gennes 1979), one readily obtains

$$\gamma = 1, \qquad \mu = d \tag{6}$$

for any d. Results (2), (3) and (6) have been found to hold for any d by Cardy (1983), and for $d = 2$ by Redner and Majid (1983); Chakrabarti and Manna (1983) obtain $\gamma = 1$ for $d = 2$.

The motivation for the use of an RG scheme in the present case, where the exact answers are known, is twofold: for one thing, it is interesting to check the precise manner in which the approximate RG results approach the exact ones; this knowledge may be of use in other situations where only approximate answers are available. Second, there is the question of whether the exponent ν obtained in a RG calculation for a directed problem is $\nu_\parallel$, $\nu_\perp$ or something else: although there are plausibility arguments favouring $\nu_\parallel$ in the case of directed percolation (Redner 1982), the question has not been fully settled (Redner and Yang (1982); see also Klein and Kinzel (1981) for a discussion of the meaning of $\xi_\perp$ in directed percolation).

In order to address the two points raised above, we generalise a PSRG approach originally introduced for undirected SAWs (de Queiroz and Chaves 1980), now allowing for directionality: to each step a fugacity p is assigned; only directed steps are allowed.

The lattice is partitioned into cells of linear size b; each of these cells is transformed, in the renormalised lattice, into a simpler cell of linear size 1. To each bond of the renormalised cell, a fugacity p' is assigned, which equals the sum of the fugacities of all walks on the primitive cell that traverse it in a given direction, starting from the corner. The non-trivial fixed point p^* of the transformation $p' = f_b(p)$ gives the RG estimate for the critical fugacity; $\nu = \ln b / \ln(\mathrm{d}p'/\mathrm{d}p)_{p^*}$ is the RG estimate for the correlation-length exponent. As is the case for SAWs and lattice animals, the critical fugacity is $p_c = 1/\mu$ (McKenzie 1976, Family 1982). From (6),

$$p_c = 1/d \tag{7}$$

in our case.

We now specialise to two dimensions, where it is possible to obtain a simple form for the RG transformation for any b. We note that the configurations we are looking for are the same as in Domany and Kinzel (1981), in the context of an anisotropic directed percolation problem (with all horizontal bonds present). The full recursion relation is found to be

$$p'_b = p^b \left[\sum_{L=0}^{b-1} \binom{L-1+b}{L} p^L \right] \equiv p^b \Sigma_b(p). \tag{8}$$

Recursion relation (8) has $p^* = \frac{1}{2}$ as a fixed point for any b. To see this, note that (a) $\Sigma_2(p) = 1 + 2p = 2$ at $p = \frac{1}{2}$; (b) it is easy to show that if $\Sigma_b(1/2) = 2^{b-1}$, then $\Sigma_{b+1}(1/2) = 2^b$. From (7), this is the exact result in two dimensions; the situation here resembles the case of isotropic bond percolation on a square lattice, where the use of self-dual cells in a RG transformation implies that the exact value of the critical concentration (1/2) is obtained for any cell size (see e.g. Oliveira *et al* 1980).

We do not know at present what lattice property (if any) similar to self-duality reflects itself in this exact result; however, we would like to point out that, in $d = 3$, the recursion relation for a $2 \times 2 \times 2$ cell is

$$p' = p^2 + 4p^3 + 6p^4 \tag{9}$$

which admits $p^* = \frac{1}{3}$ as its non-trivial fixed point; from (7), this is also the exact value for p_c in $d = 3$. This fact has no parallel in isotropic bond percolation; there, the $d = 3$ cells provide approximate results for the threshold concentration (see e.g. de Magalhães *et al* 1980).

Referring back to the two-dimensional problem, it can be shown easily that the eigenvalue of transformation (8) is given by

$$\left. \frac{\mathrm{d}p'_b}{\mathrm{d}p} \right|_{p=1/2} = 2b \left[1 - \left(\frac{1}{2} \right)^{2b} \binom{2b}{b} \right]. \tag{10}$$

From this, and using Stirling's formula $N! \simeq N^N \, \mathrm{e}^{-N} (2\pi N)^{1/2}$ $(N \to \infty)$, one obtains

$$\mathrm{d}p'_b/\mathrm{d}p|_{p=1/2} = 2b[1 - (\pi b)^{-1/2}] \qquad (b \to \infty). \tag{11}$$

Hence the RG approximation for the correlation-length exponent, ν_b, is

$$\nu_b = 1 - \ln 2 / \ln b \qquad (b \to \infty) \tag{12}$$

to leading order in inverse powers of b.

If instead of 'cell-to-bond' transformation we perform a 'cell-to-cell' renormalisation (Reynolds *et al* 1980) between cells of respective sizes b and b', the fixed point

is still at $p=\frac{1}{2}$, and the approximation for ν is

$$\nu_{b',b}=\ln(b'/b)\Big/\ln\left[\left(\frac{\mathrm{d}p'_{b'}}{\mathrm{d}p}\right)_{p=1/2}\Big/\left(\frac{\mathrm{d}p'_{b}}{\mathrm{d}p}\right)_{p=1/2}\right] \tag{13}$$

whence one obtains from (11) for $b'=b+1$ and $b\to\infty$:

$$\nu_{b+1,b}=1-1/2\sqrt{\pi b} \tag{14}$$

to leading order in inverse powers of b.

At this point, some remarks are worth making.

(i) Equation (12) is one of the few examples where the finite-size scaling hypothesis $\nu_b-\nu\sim(\ln b)^{-1}$ is analytically exhibited (see e.g. Tsallis (1982) and references therein for a discussion of this point).

(ii) The convergence of $\nu_{b+1,b}$ towards 1 is much faster than that of ν_b; this is expected on general grounds (Reynolds *et al* 1980). However, the inverse square-root form displayed above may not be of general validity: Tsallis (1982), for example, finds $\nu_{b,b-2}-\nu\sim b^{-d}$ for a majority-rule model.

(iii) The value of ν asymptotically obtained equals that of $\nu_\parallel$, thus giving support to arguments like the one of Redner (1981); also, the slow convergence shown here may, if it has a parallel in the case of directed animals, explain why the RG calculations of Redner and Yang (1982) were unable to pinpoint either $\nu_\parallel$ or $\nu_\perp$ unambiguously.

(iv) Although in the present case convergence is eventually reached, this may not always be true: in the problem of directed percolation, Kertesz and Vicsek (1980) obtained, through extrapolation of Monte Carlo results for lattices of up to 400×400 sites, a value of $\nu_\parallel=1.65$; the accepted value is 1.74.

Chakrabarti and Manna (1983) have introduced a model which is slightly different from ours; on a square lattice, they study SAWs that have a preferential orientation only along one cartesian axis, say, the vertical one; in the perpendicular direction, the walks can go either way (backward steps are of course not allowed). In analogy with the distinction made by Redner and Yang (1982) concerning lattice animals, we call their model *partially directed* SAWs, whereas ours is *fully directed* SAWs. As the difference between the two models amounts to changing the direction of the external bias (Redner and Yang 1982, Kertesz and Vicsek 1980), it is expected that both problems are in the same universality class.

Indeed, Redner and Majid (1983) find $\nu_\perp=\frac{1}{2}$ and $\nu_\parallel=1$ for partially directed SAWs on a number of two-dimensional lattices; Cardy (1983) also obtains these results from a field-theoretical approach.

However, from their analysis of numerical simulations for steps of length up to $N=14$, Chakrabarti and Manna (1983) obtain a value of $\nu=0.86\pm0.02$, which is clearly in disagreement with our results. Since their data come from exact enumeration of SAWs, the average end-to-end distance thus calculated must be sensitive to the dominant diverging length, that is, $\xi_\parallel$. In this connection, we would like to point out that the logarithmic convergence that arises in the RG treatment may be showing that properties of directed SAWs converge slowly; for example, from (10) we obtain $\nu_{14}=0.832$ whereas from (13) we obtain $\nu_{14,13}=0.919$. It is to be noted that, for ν_{14}, the smallest walk that enters in the RG calculation is of length 14 steps. Although different methods cannot be compared so directly, we feel that a RG calculation on a 14×14 cell already takes into account a significant number of configurations; even so, the result obtained for ν is rather poor. Perhaps it is this sort of effect that manifests

itself in Chakrabarti and Manna's results. See Redner and Majid (1983) for further comments on this point.

In summary, we have presented a discussion of critical properties of fully directed SAWs on hypercubic lattices. A PSRG transformation has been carried out analytically in two dimensions for any value of the scaling factor, and the finite-size scaling properties of the transformation have been explicitly exhibited. It has been shown that the RG transformation in this case is sensitive to the parallel correlation-length exponent, thus giving support to existing arguments.

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